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Cosmic Strings & Cosmic Variance

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Abstract

By using a simple analytical model based on counting random multiple impulses inflicted on photons by a network of cosmic strings we show how to construct the general q-point temperature correlation function of the Cosmic Microwave Background radiation. Our analysis is sensible specially for large angular scales where the Kaiser-Stebbins effect is dominant. Then we concentrate our study on the four-point function and in particular on its zero-lag limit, namely, the excess kurtosis parameter, for which we obtain a predicted value of $\sim 10^{-2}$. In addition, we estimate the cosmic variance for the kurtosis due to a Gaussian fluctuation field, showing its dependence on the primordial spectral index of density fluctuations n and finding agreement with previous published results for the particular case of a flat Harrison-Zel'dovich spectrum. Our value for the kurtosis compares well with previous analyses but falls below the threshold imposed by the cosmic variance when commonly accepted parameters from string simulations are considered. In particular the non-Gaussian signal is found to be inversely proportional to the scaling number of defects, as could be expected by the central limit theorem.

Subject headings: cosmic microwave background - cosmic strings

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1 Introduction

A central concept for particle physics theories attempting to unify the fundamental interactions is the concept of symmetry breaking. This symmetry breaking plays a crucial role in the Weinberg-Salam standard electroweak model (Masiero 1984) whose extraordinary success in explaining electroweak scale physics reaches a precision rarely found before in other areas of science (Koratzinos 1994). In the context of the standard Big Bang cosmological theory the spontaneous breaking of fundamental symmetries is realized as a phase transition in the early universe. Such phase transitions have several exciting cosmological consequences thus providing an important link between particle physics and cosmology.

A particularly interesting cosmological issue is the origin of structure in the universe. This structure is believed to have emerged by the gravitational growth of primordial matter fluctuations which are superposed on the smooth background required by the cosmological principle, the main assumption of the Big Bang theory.

The above mentioned link of cosmology to particle physics theories has led to the generation of two classes of theories which attempt to provide physically motivated solutions to the problem of the origin of structure in the universe. According to the one class of theories, based on inflation, primordial fluctuations arose from zero point quantum fluctuations of a scalar field during an epoch of superluminal expansion of the scale factor of the universe. These fluctuations may be shown to obey Gaussian statistics to a very high degree and to have an approximately scale invariant power spectrum.

According to the second class of theories, those based on topological defects, primordial fluctuations were produced by a superposition of seeds made of localized energy density trapped during a symmetry breaking phase transition in the early universe. Topological defects with linear geometry are known as cosmic strings and may be shown to be consistent with standard cosmology unlike their pointlike (monopoles) and planar (domain walls) counterparts which require dilution by inflation to avoid overclosing the universe. Cosmic strings are predicted to form during a phase transition in the early universe by many but not all Grand Unified Theories (GUTs).

The main elegant feature of the cosmic string model that has caused significant attention during the past decade is that the only free parameter of the model (the effective mass per unit length of the wiggly string μ) is fixed to approximately the same value from two completely independent directions. From the *microphysical* point of view the constraint $G\mu \simeq (m_{GUT}/m_P)^2 \simeq 10^{-6}$ is imposed in order that strings form during the physically realizable GUT phase transition. From the *macrophysical* point of view the same constraint arises by demanding that the string model be consistent with measurements of Cosmic Microwave Background (CMB) anisotropies and that fluctuations are strong enough for structures to form by the present time. This mass-scale ratio for $G\mu$ is actually a very attractive feature and even models of inflation have been proposed (Freese et al. 1990, Dvali et al. 1994) where observational predictions are related to similar mass scale relations.

Cosmic strings can account for the formation of large scale filaments and sheets (Vachaspati 1986; Stebbins et al. 1987; Perivolaropoulos, Brandenberger & Stebbins 1990; Vachaspati & Vilenkin 1991; Vollick 1992; Hara & Miyoshi 1993), galaxy formation at epochs $z \sim 2-3$ (Brandenberger et al. 1987) and galactic magnetic fields (Vachaspati 1992b). They also generate peculiar velocities on large scales (Vachaspati 1992a; Perivolaropoulos &

Vachaspati 1993), and are consistent with the amplitude, spectral index (Bouchet, Bennett & Stebbins 1988; Bennett, Stebbins & Bouchet 1992; Perivolaropoulos 1993a; Hindmarsh 1993) and the statistics (Gott et al. 1990; Perivolaropoulos 1993b; Moessner, Perivolaropoulos & Brandenberger 1993; Coulson et al. 1993; Luo 1994) of the CMB anisotropies measured by COBE on angular scales of order $\theta \sim 10°$.

Strings may also leave their imprint on the CMB mainly in three different ways. The best studied mechanism for producing temperature fluctuations on the CMB by cosmic strings is the Kaiser-Stebbins effect (Kaiser & Stebbins 1984; Gott 1985). According to this effect, moving long strings present between the time of recombination t_{rec} and today produce (due to their deficit angle (Vilenkin 1981)) discontinuities in the CMB temperature between photons reaching the observer through opposite sides of the string. Another mechanism for producing CMB fluctuations by cosmic strings is based on potential fluctuations on the last scattering surface (LSS). Long strings and loops present between the time of equal matter and radiation t_{eq} and the time of recombination t_{rec} induce density and velocity fluctuations to their surrounding matter. These fluctuations grow gravitationally and at t_{rec} they produce potential fluctuations on the LSS. Temperature fluctuations arise because photons have to climb out of a potential with spatially dependent depth. A third mechanism for the production of temperature anisotropies is based on the Doppler effect. Moving long strings present on the LSS drag the surrounding plasma and produce velocity fields. Thus, photons scattered for last time on these perturbed last scatterers suffer temperature fluctuations due to the Doppler effect.

It was recently shown (Perivolaropoulos 1994) how, by superposing the effects of these three mechanisms at all times from t_{rec} to today, the power spectrum of the total temperature perturbation may be obtained. It turns out that (assuming standard recombination) both Doppler and potential fluctuations at the LSS dominate over post-recombination effects on angular scales below 2°. However this is not the case for very large scales (where we will be focusing in the present paper) and this justifies our neglecting the former two sources of CMB anisotropies. The main effect of these neglected perturbations is an increase of the gaussian character of the fluctuations on small angular scales.

The main assumptions of the model were explained in (Perivolaropoulos 1993a). Here we will only review them briefly for completeness. As mentioned above, discontinuities in the temperature of the photons arise due to the peculiar nature of the spacetime around a long string which even though is *locally* flat, *globally* has the geometry of a cone with deficit angle $8\pi G\mu$. Several are the cosmological effects produced by the mere existence of this deficit angle (Shellard 1994), e.g., arcsecond-double images from GUT strings at redshifts $z \sim 1$, flatten structures from string wakes or elongated filamentary structures from slowly moving long wiggly strings and, of more relevance in our present work, post-recombination CMB anisotropy (White, Scott and Silk 1994) string induced effects (Kaiser & Stebbins 1984).

The magnitude of the discontinuity is proportional not only to the deficit angle but also to the string velocity v_s and depends on the relative orientation between the unit vector along the string $\hat{s}$ and the unit photon wave-vector $\hat{k}$. It is given by (Stebbins 1988)

$$\frac{\Delta T}{T} = \pm 4\pi G\mu v_s \gamma_s \hat{k} \cdot (\hat{v}_s \times \hat{s}) \tag{1}$$

where γ_s is the relativistic Lorentz factor and the sign changes when the string is crossed.

Also, long strings within each horizon have random velocities, positions and orientations.

We discretize the time between t_{rec} and today by a set of N Hubble time-steps t_i such that $t_{i+1} = t_i\, \delta$, i.e., the horizon gets multiplied by δ in each time-step. For $z_{rec} \sim 1400$ we have $N \simeq \log_\delta[(1400)^{3/2}]$.

In the frame of the multiple impulse approximation the effect of the string network on a photon beam is just the linear superposition of the individual effects, taking into account compensation (Traschen et al. 1986; Veeraraghavan & Stebbins 1990; Magueijo 1992), that is, only those strings within a horizon distance from the beam inflict perturbations to the photons.

In the following section we show how to construct the general q-point function of CMB anisotropies at large angular scales produced through the Kaiser-Stebbins effect. Explicit calculations are performed for the four-point function and its zero-lag limit, the kurtosis. Next, we calculate the (cosmic) variance for the kurtosis assuming Gaussian statistics for arbitrary value of the spectral index and compare it with the string predicted value (section 3). Finally, in section 4 we briefly discuss our results.

2 The Four–Point Temperature Correlation Function

According to the previous description, the total temperature shift in the $\hat{\gamma}$ direction due to the Kaiser-Stebbins effect on the microwave photons between the time of recombination and today may be written as

$$\frac{\Delta T}{T}(\hat{\gamma}) = 4\pi G\mu v_s \gamma_s \sum_{n=1}^{N} \sum_{m=1}^{M} \beta^{mn}(\hat{\gamma}) \tag{2}$$

where $\beta^{mn}(\hat{\gamma})$ gives us information about the velocity v^{mn} and orientation s^{mn} of the mth string at the nth Hubble time-step and may be cast as $\beta^{mn}(\hat{\gamma}) = \hat{\gamma} \cdot \hat{R}^{mn}$, with $\hat{R}^{mn} = v^{mn} \times s^{mn} = (\sin\theta^{mn}\cos\phi^{mn}, \sin\theta^{mn}\sin\phi^{mn}, \cos\theta^{mn})$ a unit vector whose direction varies randomly according to the also random orientations and velocities in the string network. In Eq.(2), M denotes the mean number of strings per horizon scale, obtained from simulations (Allen & Shellard 1990, Bennett & Bouchet 1988) to be of order $M \sim 10$.

Let us now study the correlations in temperature anisotropies. We are focusing here in the stringy-perturbation inflicted on photons after the time of recombination on their way to us. Photons coming from different directions of the sky share a common history that may be long or short depending on their angular separation. We therefore take $\theta_{t_p} \simeq z(t_p)^{-1/2}$ to be the angular size of the horizon at Hubble time-step t_p ($1 \leq p \leq N$). The kicks on two different photon beams separated by an angular scale $\alpha_{12} = \arccos(\hat{\gamma}_1 \cdot \hat{\gamma}_2)$ greater than θ_{t_p} will be uncorrelated for the time-step t_p (no common history up until $\sim t_p$) but will eventually become correlated (and begin sharing a common past) afterwards when the horizon increases, say, at $t_{p'}$ when $\theta_{t_{p'}} \gtrsim \alpha_{12}$, i.e., when α_{12} *fits* in the horizon scale.

Much in the same way, kicks inflicted on three photon beams will be uncorrelated if at time t any one of the three angles between any two directions (say, α_{12}, α_{23}, α_{13}) is greater than θ_t, the size of the horizon at time t. So, in this case, we will be summing (cf. Eq.(2)) over those Hubble time-steps n greater than p, where p is the time-step when the condition

$\theta_{t_p} = \text{Max}[\alpha_{12}, \alpha_{23}, \alpha_{13}]$ is satisfied. The same argument could be extended to any number of photon beams (and therefore for the computation of the q–point function of temperature anisotropies)[2].

Let us put the above considerations on more quantitative grounds, by writing the mean q-point correlation function. Using Eq.(2) we may express it as

$$< \frac{\Delta T}{T}(\hat{\gamma}_1)...\frac{\Delta T}{T}(\hat{\gamma}_q) >= \xi^q \sum_{n_1,...,n_q=1}^{N} \sum_{m_1,...,m_q=1}^{M} < \beta_1^{m_1 n_1}...\beta_q^{m_q n_q} > \tag{3}$$

where $\hat{\gamma}_1...\hat{\gamma}_q$ are unit vectors denoting directions in the sky and where $\beta_j^{m_j n_j} = \hat{\gamma}_j \cdot \hat{R}^{m_j n_j}$ with $\hat{\gamma}_j = (\sin\theta_j \cos\phi_j, \sin\theta_j \sin\phi_j, cos\theta_j)$. In the previous equation we defined $\xi \equiv 4\pi G\mu v_s \gamma_s$. For simplicity we may always choose a coordinate system such that $\theta_1 = 0$, $\phi_1 = 0$ (i.e., $\hat{\gamma}_1$ lies on the $\hat{z}$ axis) and $\phi_2 = \pi/2$. The seemingly complicated sum (3) is in practice fairly simple to calculate because of the large number of terms that vanish due to lack of correlation, after the average is taken. The calculation proceeds by first splitting the product $\beta_1^{m_1 n_1}...\beta_q^{m_q n_q}$ into all possible sub-products that correspond to correlated kicks (i.e., $\beta^{m_j n_j}$'s with the same pair (m_j, n_j)) at each expansion Hubble-step and then evaluating the ensemble average of each sub-product by integration over all directions of $\hat{R}$. To illustrate this technique we evaluate the two, three and four-point functions below.

The calculation of *the two-point function* has been performed in (Perivolaropoulos 1993a) but we briefly repeat it here for clarity and completeness. Having a correlated pair of beams in $\hat{\gamma}_1$ and $\hat{\gamma}_2$ directions from a particular time-step p onwards means simply that $\hat{R}^{m_1 n_1} = \hat{R}^{m_2 n_2} \Leftrightarrow n \equiv n_1 = n_2 > p$, $m \equiv m_1 = m_2$; otherwise the $\hat{R}$'s remain uncorrelated and there is no contribution to the mean two-point function. Therefore we will have

$$\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2) = \xi^2 \sum_{n=p}^{N} \sum_{m=1}^{M} (\hat{\gamma}_1 \cdot \hat{R}^{mn})(\hat{\gamma}_2 \cdot \hat{R}^{mn}) \tag{4}$$

where we wrote just the correlated part on an angular scale $\alpha_{12} = \arccos(\hat{\gamma}_1 \cdot \hat{\gamma}_2)$. The uncorrelated part on this scale will vanish when performing the ensemble average $\langle\cdot\rangle$. Thus the mean two-point function may be written as

$$< \frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2) >= \xi^2 < \beta_1 \beta_2 > N_{cor}(\alpha_{12}) \tag{5}$$

where

$$N_{cor}(\alpha_{12}) \equiv M[N - 3\log_\delta(1 + \frac{\alpha_{12}}{\theta_{t_{rec}}})] \tag{6}$$

is the number of correlated 'kicks' inflicted on a scale α_{12} after this scale enters the horizon. θ_{rec} is the angular scale of the horizon at the time of recombination. Since we may always take

$$\beta_1 = (0, 0, 1) \cdot \hat{R} \qquad\qquad \beta_2 = (\sin\alpha_{12}, 0, \cos\alpha_{12}) \cdot \hat{R} \tag{7}$$

[2]A general expression (suitable to whatever angular scale and to whatever source of temperature fluctuations) for the three-point correlation function was given in (Gangui, Lucchin, Matarrese and Mollerach 1994). Although of much more complexity, similar analysis may be done for the four-point function and a completely general expression in terms of the angular trispectrum may be found (Gangui & Perivolaropoulos 1994).

with $\hat{R} = (\sin\theta\cos\phi, \sin\theta\sin\phi, cos\theta)$, $< \frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2) >$ may be calculated by integrating over (θ, ϕ) and dividing by 4π. The result is

$$< \frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2) >= \xi^2 \frac{\cos\alpha_{12}}{3} N_{cor}(\alpha_{12}) \tag{8}$$

which may be shown (Perivolaropoulos 1993a) to lead to a slightly tilted scale invariant spectrum on large angular scales.

The *three-point function* may be obtained in a similar way. However, the superimposed kernels of the distribution turn out to be symmetric with respect to positive and negative perturbations and therefore no mean value for the three-point correlation function arises (in particular also the skewness is zero). On the other hand the four-point function is easily found and a mean value for the excess kurtosis parameter (Gangui 1994b) may be predicted, as we show below.

In the case of the *four-point function* we could find terms where (for a particular time-step) the photon beams in directions $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\gamma}_3$ are all correlated amongst themselves but not the beam $\hat{\gamma}_4$, which may be taken sufficiently far apart from the other three directions. In such a case we have

$$\langle\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\frac{\Delta T}{T}(\hat{\gamma}_3)\frac{\Delta T}{T}(\hat{\gamma}_4)\rangle \longrightarrow \langle\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\frac{\Delta T}{T}(\hat{\gamma}_3)\rangle\ \langle\frac{\Delta T}{T}(\hat{\gamma}_4)\rangle \tag{9}$$

and (cf. (Perivolaropoulos 1993a)) this contribution vanishes.

Another possible configuration we could encounter is the one in which the beams are correlated two by two but no correlation exists between the pairs (for one particular time-step). This yields three distinct possible outcomes, e.g., $\langle\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\rangle\,\langle\frac{\Delta T}{T}(\hat{\gamma}_3)\frac{\Delta T}{T}(\hat{\gamma}_4)\rangle$, and the other two obvious combinations.

The last possibility is having all four photon beams fully correlated amongst themselves and this yields $\langle\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\frac{\Delta T}{T}(\hat{\gamma}_3)\frac{\Delta T}{T}(\hat{\gamma}_4)\rangle_c$, where the subscript c stands for the *connected* part.

In a way completely analogous to that for the two-point function, the correlated part for the combination of four beams gives

$$\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\frac{\Delta T}{T}(\hat{\gamma}_3)\frac{\Delta T}{T}(\hat{\gamma}_4) = \xi^4 \sum_{n=p}^{N}\sum_{m=1}^{M}(\hat{\gamma}_1\cdot\hat{R}^{mn})(\hat{\gamma}_2\cdot\hat{R}^{mn})(\hat{\gamma}_3\cdot\hat{R}^{mn})(\hat{\gamma}_4\cdot\hat{R}^{mn}) \tag{10}$$

Now we are ready to write the full mean four–point function as

$$\begin{aligned}
&\langle\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\frac{\Delta T}{T}(\hat{\gamma}_3)\frac{\Delta T}{T}(\hat{\gamma}_4)\rangle \\
&\quad=\Big[\langle\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\rangle\langle\frac{\Delta T}{T}(\hat{\gamma}_3)\frac{\Delta T}{T}(\hat{\gamma}_4)\rangle+2\text{terms}\Big]+\langle\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\frac{\Delta T}{T}(\hat{\gamma}_3)\frac{\Delta T}{T}(\hat{\gamma}_4)\rangle_c \\
&\quad= 3\left[\frac{1}{3}\xi^2 N_{cor}(\theta)\cos(\theta)\right]^2 + \langle\frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\frac{\Delta T}{T}(\hat{\gamma}_3)\frac{\Delta T}{T}(\hat{\gamma}_4)\rangle_c
\end{aligned} \tag{11}$$

where for simplicity we wrote this equation using the same scale θ for all directions on the sky (first term after the equality sign); after all, we will be interested in the zero lag limit, in which case we will have $\theta \to 0$. By using *Mathematica* (Wolfram 1991) it is simple to find that the second term includes just the sum of twenty-one combinations of trigonometric

functions depending on θ_i, ϕ_i, with $i = 1, 2, 3, 4$, the spherical angles for the directions $\hat{\gamma}_i$ on the sky (cf. Eq.(10)). These terms are the only ones which contribute non-vanishingly after the integration over the angles θ^{mn}, ϕ^{mn} is performed (assuming ergodicity).

When we take the zero lag limit (i.e., aiming for the kurtosis) the above expression gets largely simplified. After normalizing by the squared of the variance $\sigma^4 = [\frac{1}{3}\xi^2 N_{cor}(0)]^2$ and subtracting the disconnected part we get the excess kurtosis parameter

$$\mathcal{K} = \frac{1}{\sigma^4}\langle \frac{\Delta T}{T}^4 (\hat{\gamma}_1)\rangle - 3 = \frac{9}{5MN} \simeq (1.125\ ^{+0.675}_{-0.458}) \times 10^{-2} \qquad (12)$$

where we took $M \simeq 10$ and $\delta = 2$ implying $N \simeq 16$. Quoted errors take into account possible variation of δ in the range $1.5 \leq \delta \leq 3$.

Note that for values of the scaling solution M increasing (large number of seeds) the actually small non-Gaussian signal $\mathcal{K}$ gets further depressed, as it could be expected from the Central Limit Theorem.

3 The r.m.s. excess kurtosis of a Gaussian field

The previous section was devoted to the computation of $\mathcal{K}$, the excess kurtosis parameter, as predicted by a simple analytical model in the framework of the Cosmic String scenario. We might ask whether this particular non-Gaussian signal has any chance of actually being unveiled by current anisotropy experiments. Needless to say, this could provide a significant probe of the structure of the relic radiation and furthermore give us a hint on the possible sources of the primordial perturbations that, after non-linear evolution, realize the large scale structure of the universe as we currently see it.

However, as it was realized some time ago (Scaramella & Vittorio 1991; Abbott & Wise 1984; Srednicki 1993; Gangui 1994a), the mere detection of a non-zero higher order correlation function (e.g., the four-point function) or its zero lag limit (e.g., the kurtosis) cannot be directly interpreted as a signal for intrinsically non-Gaussian perturbations. In order to tell whether or not a particular measured value for the kurtosis constitutes a significant evidence of a departure from Gaussian statistics, we need to know the amplitude of the non-Gaussian pattern produced by a *Gaussian* perturbation field. Namely, we need to know the r.m.s. excess kurtosis of a Gaussian field.

Let us begin with some basics. Let us denote the kurtosis $K \equiv \int \frac{d\Omega_{\hat{\gamma}}}{4\pi} \frac{\Delta T}{T}^4(\hat{\gamma})$ and assume that the underlying statistics is Gaussian, namely, that the multipole coefficients a_ℓ^m are Gaussian distributed. Thus, the ensemble average for the kurtosis is given by the well-known formula: $\langle K\rangle = 3\sigma^4$, where σ^2 is the mean two–point function at zero lag, i.e., the CMB variance as given by

$$\sigma^2 \equiv \langle C_2(0)\rangle = \frac{1}{4\pi}\sum_\ell \frac{(2\ell+1)}{5}\mathcal{Q}^2\mathcal{C}_\ell\mathcal{W}_\ell^2 \qquad (13)$$

where the $\mathcal{C}_\ell$ coefficients (here normalized to $\mathcal{C}_{\ell=2} = 1$) are also dependent on the value for the primordial spectral index of density fluctuations n and are given by the usual expression

in terms of Gamma functions (Bond & Efstathiou 1987; Fabbri, Lucchin & Matarrese 1987)

$$\mathcal{C}_\ell = \frac{\Gamma(\ell + \frac{n}{2} - \frac{1}{2})\Gamma\left(\frac{9}{2} - \frac{n}{2}\right)}{\Gamma\left(\ell + \frac{5}{2} - \frac{n}{2}\right)\Gamma(\frac{3}{2} + \frac{n}{2})} \tag{14}$$

$\mathcal{Q} = \langle Q_2^2 \rangle^{1/2}$ is the *rms* quadrupole, simply related to the quantity Q_{rms-PS} defined in (Smoot et al. 1992; Bennett et al. 1994) by $\mathcal{Q} = \sqrt{4\pi} Q_{rms-PS}/T_0$, with mean temperature $T_0 = 2.726 \pm 0.01K$ (Mather et al. 1994). In the previous expression $\mathcal{W}_\ell$ represents the window function of the specific experiment. Setting $\mathcal{W}_0 = \mathcal{W}_1 = 0$ automatically accounts for both monopole and dipole subtraction. In the particular case of the COBE experimental setup we have, for $\ell \geq 2$, $\mathcal{W}_\ell \simeq \exp\left[-\frac{1}{2}\ell(\ell+1)(3.2^\circ)^2\right]$, where 3.2° is the dispersion of the antenna–beam profile, which measures the angular response of the detector (e.g. Wright et al. 1992). Sometimes the quadrupole term is also subtracted from the maps and in that case we also set $\mathcal{W}_2 = 0$.

However, $\langle K \rangle$ is just the mean value of the distribution and therefore we cannot know its real value but within some error bars. In order to find out how probable it is to get this value after a set of experiments (observations) is performed, we need to know the variance of the distribution for K. In other words, we ought to know how peaked the distribution is around its mean value $\langle K \rangle$. The width of the distribution is commonly parameterized by what is called the cosmic variance of the kurtosis

$$\sigma_{CV}^2 = \langle K^2 \rangle - \langle K \rangle^2 \tag{15}$$

It is precisely this quantity what attaches theoretical error bars to the actual value for the kurtosis. Therefore, we may heuristically express the effect of σ_{CV}^2 on K_{Gauss} as follows: $K_{Gauss} \simeq \langle K \rangle \pm \sigma_{CV}$, at one sigma level (a good approximation in the case of a narrow peak). Re-arranging factors we may write this expression in a way convenient for comparing it with $\mathcal{K}$ as follows

$$\mathcal{K}_{CV} = \frac{K_{Gauss}}{\sigma^4} - 3 \simeq \pm \frac{\sigma_{CV}}{\sigma^4} \tag{16}$$

where $\mathcal{K}_{CV}$ is the excess kurtosis parameter (assuming Gaussian statistics) purely due to the cosmic variance. Not only is $\mathcal{K}_{CV}$ in general non-zero, but its magnitude increases with the theoretical uncertainty (σ_{CV}) due to the limitation induced by our impossibility of making measurements in more than one Universe.

This gives a fundamental threshold that must be overcome by any measurable kurtosis parameter in order for us to be able to distinguish the primordial non-Gaussian signal from the theoretical noise in which it is embedded. In other words, unless our predicted value for $\mathcal{K}$ *exceeds* $\mathcal{K}_{CV}$ we will not be able to tell confidently that any measured value of $\mathcal{K}$ is due to primordial non-Gaussianities.

Now that we know the expression for the cosmic variance of the kurtosis, let us calculate it explicitly. We begin by calculating $\langle K^2 \rangle$ as follows

$$\langle K^2 \rangle = \int \frac{d\Omega_{\hat{\gamma}_1}}{4\pi} \int \frac{d\Omega_{\hat{\gamma}_2}}{4\pi} \langle \frac{\Delta T}{T}^4(\hat{\gamma}_1) \frac{\Delta T}{T}^4(\hat{\gamma}_2) \rangle \tag{17}$$

By assuming Gaussian statistics for the temperature perturbations $\frac{\Delta T}{T}(\hat{\gamma})$ we may make use of standard combinatoric relations, and get

$$\langle \frac{\Delta T}{T}^4(\hat{\gamma}_1)\frac{\Delta T}{T}^4(\hat{\gamma}_2)\rangle = 9\,\langle \frac{\Delta T}{T}^2(\hat{\gamma}_1)\rangle^2 \langle \frac{\Delta T}{T}^2(\hat{\gamma}_2)\rangle^2$$
$$+\,72\,\langle \frac{\Delta T}{T}^2(\hat{\gamma}_1)\rangle\langle \frac{\Delta T}{T}^2(\hat{\gamma}_2)\rangle\langle \frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\rangle^2 + 24\,\langle \frac{\Delta T}{T}(\hat{\gamma}_1)\frac{\Delta T}{T}(\hat{\gamma}_2)\rangle^4 \tag{18}$$

As we know, the ensemble averages are rotationally invariant and therefore $\langle \frac{\Delta T}{T}^2(\hat{\gamma}_1)\rangle = \langle C_2(0)\rangle \equiv \sigma^2$ is independent of the direction $\hat{\gamma}_1$. Plug this last equation into Eq.(17) and we get

$$\langle K^2\rangle = 9\sigma^8 + 36\sigma^4 \int_{-1}^{1} d\cos\alpha\, \langle C_2(\alpha)\rangle^2 + 12\int_{-1}^{1} d\cos\alpha\, \langle C_2(\alpha)\rangle^4. \tag{19}$$

The above integrals may be solved numerically. Then, using this result in the expression for σ_{CV}^2 we get $\mathcal{K}_{CV}$, the value for the excess kurtosis parameter of a Gaussian field.

It is also instructive to look at Eq.(19) in some more detail, so that the actual dependence on the spectral index becomes clear. By expanding the mean two-point correlation functions within this expression in terms of spherical harmonics and after some long but otherwise straightforward algebra we find

$$\mathcal{K}_{CV} = \Bigg[72\,\frac{\sum_\ell (2\ell+1)\mathcal{C}_\ell^2 \mathcal{W}_\ell^4}{[\sum_\ell (2\ell+1)\mathcal{C}_\ell \mathcal{W}_\ell^2]^2}$$
$$+\; 24\,\frac{\{\prod_{i=1}^4 \sum_{\ell_i}\sum_{m_i=-\ell_i}^{\ell_i} \mathcal{C}_{\ell_i}\mathcal{W}_{\ell_i}^2\}\left(\sum_L 4\pi\, \bar{\mathcal{H}}_{\ell_1,\,\ell_2,\;L}^{m_1,m_2,m_3+m_4}\bar{\mathcal{H}}_{\ell_3,\,\ell_4,\;L}^{m_3,m_4,-m_3-m_4}\right)^2}{[\sum_\ell (2\ell+1)\mathcal{C}_\ell \mathcal{W}_\ell^2]^4}\Bigg]^{1/2} \tag{20}$$

where the coefficients $\bar{\mathcal{H}}_{\ell_1,\ell_2,\ell_3}^{m_1,m_2,m_3} \equiv \int d\Omega_{\hat{\gamma}} Y_{\ell_1}^{m_1}(\hat{\gamma})Y_{\ell_2}^{m_1}(\hat{\gamma})Y_{\ell_3}^{m_3}(\hat{\gamma})$, which can be easily expressed in terms of Clebsch–Gordan coefficients (Messiah 1976), are only non–zero if the indices ℓ_i, m_i $(i=1,2,3)$ fulfill the relations: $|\ell_j - \ell_k| \le \ell_i \le |\ell_j + \ell_k|$, $\ell_1+\ell_2+\ell_3 = even$ and $m_1+m_2+m_3 = 0$. In the above equation the n-dependence is hidden inside the multipole coefficients $\mathcal{C}_\ell$ (cf. Eq.(14)).

Eq.(20) shows an analytic expression for computing $\mathcal{K}_{CV}$ which, in turn, represents a fundamental threshold for any given non-Gaussian signal. For interesting values of the spectral index (say, between $0.8 \lesssim n \lesssim 1.3$) we find no important variation in $\mathcal{K}_{CV}$, being its value consistent with the Monte-Carlo simulations performed by Scaramella & Vittorio 1991 (SV91 hereafter) , see below. These authors concentrated on a Harrison-Zel'dovich spectrum, considered also the quadrupole contribution and took a slightly different dispersion width for the window function (3.0° in their simulations).

We coded an IBM RISC 6000 for solving Eq.(20) numerically. We first calculated $\mathcal{K}_{CV}$ for a somewhat reduced range for the multipole index ℓ with quadrupole subtracted, $3 \le \ell \le 5$, so that we were able to test the result with that obtained by using *Mathematica*. We found perfect agreement and a value $\mathcal{K}_{CV} \simeq 1.85$ (for $n=1$). We may even go further and include the quadrupole, i.e. $2 \le \ell \le 5$, and in this case we get $\mathcal{K}_{CV} \simeq 1.92$. This slightly larger value obtained after including $\ell = 2$ is not a new feature and simply reflects the intrinsic

theoretical uncertainty of the lowest order multipoles; see e.g. (Gangui et al. 1994) for a similar situation in the case of the skewness.

Of course $\mathcal{K}_{CV} \simeq 1.85$ (or 1.92 with quadrupole) are still a factor 6 above the value ~ 0.3 found in SV91 (where they included the quadrupole). Our analytical analysis expresses $\mathcal{K}_{CV}$ as a ratio of averages (rather than the average of a ratio as in SV91) and therefore exact agreement should not be expected. But still the main reason for the discrepancy lies in the small range for the multipole index ℓ. We therefore increased the value of ℓ_{max} and checked that $\mathcal{K}_{CV}$ monotonically became smaller and smaller. When we were in the range $3 \leq \ell \leq 10$ we got $\mathcal{K}_{CV} \simeq 1.16$; instead, for $3 \leq \ell \leq 15$ we got $\mathcal{K}_{CV} \simeq 0.98$ (for $n = 1$) –clearly a sensible decrease. CPU–time limitations prevent us from carrying out the numerical computations for larger values of ℓ_{max} (usually ℓ_{max} is chosen of order 30, i.e., beyond those values of ℓ where the exponential suppression of the $\mathcal{W}_\ell$ makes higher ℓ contribution to the sums negligible), but still we expect $\mathcal{K}_{CV}$ to keep on steadily decreasing (we checked that the decrease in $\mathcal{K}_{CV}$ was tinier as ℓ_{max} got larger). This, together with some previous experience in similar computations (Gangui et al. 1994) makes us believe that in the case appropriately large values for ℓ_{max} were used , the SV91 value above mentioned would be attained (taking into account the quadrupole subtraction, of course).

In addition, we checked that a different value of the spectral index n does not change the essence of the above considerations. We explored the cosmologically interesting range $0.8 \leq n \leq 1.3$ and got values $1.22 \geq \mathcal{K}_{CV} \geq 1.08$ (taking $3 \leq \ell \leq 10$) and values $1.05 \geq \mathcal{K}_{CV} \geq 0.88$ (taking $3 \leq \ell \leq 15$). We plot $\mathcal{K}_{CV}$ versus the spectral index for $\ell_{max} = 15$ in Fig.1.

Note the small rate of variation of $\mathcal{K}_{CV}$ with n and that, as expected, $\mathcal{K}_{CV}$ takes larger values for smaller spectral indexes. This is clearly due to the fact that a small n generates more power on large scales (i.e., small ℓ) and precisely these scales are the ones that contribute the most to the cosmic variance of the kurtosis field.

4 Discussion

In the present paper we showed how to implement the multiple impulse approximation for perturbations on a photon beam (stemming from the effect of the string network) in the actual construction of higher order correlations for the CMB anisotropies. We then focused on the four-point function and on the excess kurtosis parameter, finding for the latter a value $\mathcal{K} \sim 10^{-2}$.

We also calculated explicitly the rms excess kurtosis $\mathcal{K}_{CV}$ predicted to exist even for a Gaussian underlying field and showed its dependence on the primordial spectral index of density fluctuations. This constitutes the main source of theoretical uncertainty at COBE scales. In fact, the cosmic string signature that might have been observable is actually blurred in the cosmic variance mist reigning at very large scales.

Nevertheless, there is still a chance of getting a string-characteristic angular dependence from the study of the mean four-point correlation function by exploiting the particular geometries deriving from it; namely, its collapsed cases (where some of the five independent angles are taken to be zero) or from particular choices for these angles (as in the case of

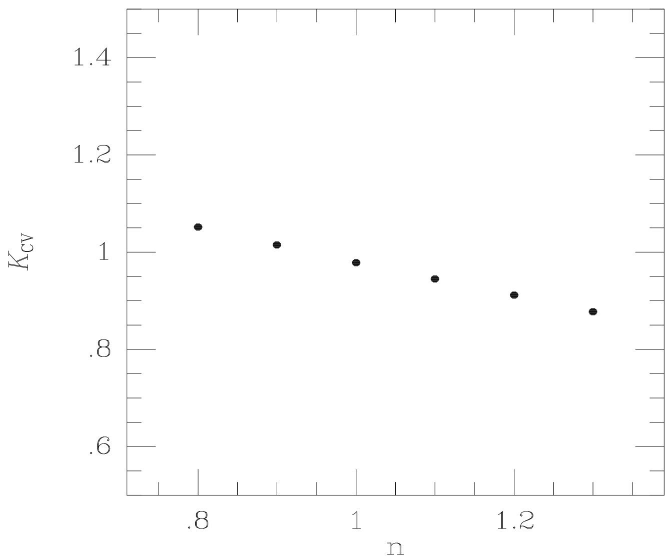

Figure 1: Excess Kurtosis parameter of a Gaussian temperature fluctuation field as function of the spectral index.

taking all angles equal).

A preliminary analysis (Gangui & Perivolaropoulos 1994) making use of just one non-vanishing angle in a collapsed configuration as the one mentioned above shows a potentially interesting effect that could eventually increase notably the small non-Gaussian signal, and suggests that this is indeed a subject worth of further investigation. Some of these alternatives are presently under study and we expect to report progress on this subject in a future publication.

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